Recitation 9

Min Kim

April 14, 2021

Outline

- Class review
- Selected homework questions
- Log-linearization
- Solving expectational difference equation

Class review

- We have been considering flexible price eqbm as the benchmark SS
  - Is it the efficient eqbm that we would like to have?
  - No, there is a real friction (monopolistic competition) making it inefficient
- Three sources of inefficiency
  i. output gap: markup variation
  ii. inflation: misallocation of labor due to price dispersion
  iii. real distortion: monopolistic competition
- Special case: no real friction (e.g. due to subsidy in goods market),
  - natural output is the efficient level of output
  - interest rule closing inflation gap stabilizes output gap: Divine Coincidence

7.12.

Consider the model of Section 7.8. Suppose, however, that monetary policy responds to current inflation and output: $r_{\mathrm{t}} = \phi_{\pi}\pi_t + \phi_y y_{\mathrm{t}} + u_{\mathrm{t}}^{\mathrm{MP}}$.

(a) For the case of white-noise disturbances, find expressions analogous to (7.92)-(7.94). What are the effects of an unfavorable inflation shock in this case?

(b) Describe how you would solve this model using the method of undetermined coefficients (but do not actually solve it).

7.12 Answer

Notice that now the interest rule depends on the current inflation and output

The eqbm equations are

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\theta} r_t + u_t^{IS}$$
$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + u_t^{\pi}$$
$$r_t = \phi_\pi \pi_t + \phi_y y_\mathrm{t} + u_\mathrm{t}^{\mathrm{MP}}$$

After a positive shock of u_t^{π},

- r_t will now go up contemporaniously, decreasing y_t and mitigating the effect on π_t

We have the same 3 state and 2 forward-looking vars: r_t is (again) just a static variable

- One can use the same guess (7.95)-(7.96) for undetermined coefficient method

Consider the case of $u_t^{IS} = u_t^{MP} = 0$

$$
\begin{aligned}
y_t &= -\frac{1}{\theta} r_t \\
\pi_t &= \kappa y_t + u_t^{\pi} \\
r_t &= \phi_\pi \pi_t + \phi_y y_\mathrm{t}
\end{aligned}
$$

Substituting r_t, we get

$$
\begin{aligned}
y_t &= \frac{-\phi_\pi}{\theta + \phi_y + \kappa\phi_\pi} u_t^{\pi} \\
\pi_t &= \frac{\theta + \phi_y}{\theta + \phi_y + \kappa\phi_\pi} u_t^{\pi} \\
r_t &= \frac{\phi_\pi(\theta + \phi_y) - \phi_y\phi_\pi}{\theta + \phi_y + \kappa\phi_\pi} u_t^{\pi}
\end{aligned}
$$

Let's check this and compare it with the baseline case in `nk4.mod`

2.1. Optimality conditions under nonseparable leisure

Derive the log-linearized optimality conditions of the household problem under the following specification of the period utility function:

$$U\left(C_t, N_t\right) = \frac{\left[C_t\left(1 - N_t\right)^{\nu}\right]^{1-\sigma} - 1}{1 - \sigma}$$

2.1 Answer

Recall that the Euler equation is

$$U_{c,t} = \beta \mathbb{E}_t U_{c,t+1} R_t$$

What is R_t the real interest rate in our case?

2.1 Answer

Recall that the Euler equation is

$$U_{c,t} = \beta \mathbb{E}_t U_{c,t+1} R_t$$

What is R_t the real interest rate in our case?

$$R_t = \frac{P_t}{P_{t+1}} \frac{1}{Q_t}$$

Let's first take log on both sides of the equation

$$\log(U_{c,t}) = \log(\beta) + \log(U_{c,t+1}) + \log(R_t)$$

and then linearize it by Taylor expansion around the SS

- LHS:

$$\log(\bar{U}_c) + \frac{\bar{U}_{cc}}{\bar{U}_c}(c_t - \bar{c})$$

- RHS:

$$\log(\beta) + \log(\bar{U}_c) + \log(\bar{R}) + \frac{\bar{U}_{cc}}{\bar{U}_c}(c_{t+1} - \bar{c}) + \frac{1}{\bar{R}}\left(R_t - \bar{R}\right)$$

ignoring the errors

Hence,

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} + \left(\frac{\bar{U}_c}{\bar{U}_{cc} \bar{c}} \right) \hat{R}_t$$

and

$$\begin{aligned} \hat{R}_t &\approx \log(R_t) - \log(\bar{R}) \\ &= r_t - \log(1/\beta) \end{aligned}$$

All we have left is to find the elasticity of intertemporal substitution of consumption

- which is $1/\sigma$ as in the textbook baseline case

Thus,

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} - \rho)$$

More on log-linearization (from professor's notes in earlier years)

In general, linearization of $Z_t = f\left(X_t, Y_t\right)$ around $\bar{Z} = f(\bar{X}, \bar{Y})$ is

$$z_t = \eta_f^x x_t + \eta_f^y y_t$$

where $\eta_f^x = \frac{\bar{X} f_1(\bar{X},\bar{Y})}{f(\bar{X},\bar{Y})}, \eta_f^y = \frac{\bar{Y} f_2(\bar{X},\bar{Y})}{f(\bar{X},\bar{Y})}$

Useful tricks

- Sum: $Z_t = X_t + Y_t$, then $z_t = \frac{\bar{X}}{\bar{X}+\bar{Y}} x_t + \frac{\bar{Y}}{\bar{X}+\bar{Y}} y_t$
- product: $Z_t = X_t Y_t$, then $z_t = x_t + y_t$
- Ratio: $Z_t = X_t / Y_t$, then $z_t = x_t - y_t$
- Composition: $Z_t = f(Y_t)$ and $Y_t = g(X_t)$, then $z_t = \eta_f^y \eta_g^x x_t$

✏️ **Exercise**: $Y_t = (a X_t^\theta + (1-a) Z_t^\theta)^{\gamma/\theta}$, then $y_t = \gamma[a \bar{X}^\theta x_t + (1-a) \bar{Z}^\theta z_t] / \bar{Y}^{\theta/\gamma}$

Solving expectational difference equation

Consider

$$y_t = a\mathbb{E}_t y_{t+1} + bx_t$$

where a and b are non-zero scalers and x_t is a stationary exogenous process

- What is the steady state?

Two cases

1. $|a| < 1$
2. $|a| > 1$

1. $|a| < 1$

1. Brute force

Note that $y_{t+1} = a\mathbb{E}_t y_{t+2} + bx_{t+1}$

$$
\begin{aligned}
y_t &= a\mathbb{E}_t[a\mathbb{E}_t y_{t+2} + bx_{t+1}] + bx_t \\
&= a^2\mathbb{E}_t\mathbb{E}_{t+1} y_{t+2} + ab\mathbb{E}_t x_{t+1} + bx_t \\
&= a^2\mathbb{E}_t y_{t+2} + ab\mathbb{E}_t x_{t+1} + bx_t \\
&\vdots \\
&= \lim_{T\to\infty} a^{T+1}\mathbb{E}_t y_{t+T+1} + b\lim_{T\to\infty}\sum_{k=0}^{T} a^k\mathbb{E}_t x_{t+T}
\end{aligned}
$$

- Would this converge?
    - Yes! $y_t = b\sum_{k=0}^{\infty} a^k\mathbb{E}_t x_{t+k}$

1. $|a| < 1$

2. Lag operator

Note that we can rewrite the equation as

$$
\begin{aligned}
y_t &= aL^{-1}y_t + bx_t \\
&= (1 - aL^{-1})^{-1}bx_t \\
&= b\sum_{k=0}^{\infty} a^k L^{-k} x_t \\
&= b\sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k}
\end{aligned}
$$

2. $|a| < 1$

In this case, we have infinite solutions. Note that $\mathbb{E}_t y_{t+1} = \frac{1}{a} y_t - \frac{b}{a} x_t$. One solution is

$$y_t = \frac{1}{a} y_{t-1} - \frac{b}{a} x_{t-1} + \varepsilon_t$$

where ε_t is iid and $\mathbb{E}_t \varepsilon_t = 0$

2.2. Alternative interest rules for the classical economy

Consider the classical economy described in the text, with equilibrium conditions

$$y_t = \mathbb{E}_t \{y_{t+1}\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \{\pi_{t+1}\} - \rho\right)$$

and

$$\begin{aligned} r_t &\equiv i_t - \mathbb{E}_t \{\pi_{t+1}\} \\ &= \rho + \sigma \mathbb{E}_t \{\Delta y_{t+1}\} \end{aligned}$$

where y_t and, hence, r_t, are determined independently of monetary policy. Analyze the implications of the following alternative monetary policy rules.

When relevant, assume that the money demand function takes the form

$$m_t - p_t = y_t - \eta i_t + \varepsilon_t^m$$

where ε_t^m is a stochastic money demand disturbance.

2.2a

Rule #1: Strict inflation targeting.

i. Derive an interest rate rule that guarantees full stabilization of inflation, i.e., $\pi_t = \pi^*$ for all t where π^* is an inflation target.

ii. Determine the behavior of money growth implied by that rule.

iii. Explain why a policy characterized by a constant rate of money growth $\Delta m_t = \pi^*$ will generally not succeed in stabilizing inflation in that economy.

2.2a Answer

i. Inspired by the rule in section 2.4.2, our guess is $i_t - i = \phi_\pi(\pi_t - \pi^*) + v_t$ with $\phi_\pi > 1$

From the Fisher equation: $i_t - i = \mathbb{E}_t(\pi_{t+1} - \pi^*) + r_t - \rho$ as a deviation from the SS

$$\phi_\pi \hat{\pi}_t + v_t = \mathbb{E}_t \hat{\pi}_{t+1} + \hat{r}_t$$

which is

$$\left(1 - \frac{1}{\phi_\pi} L^{-1}\right) \hat{\pi}_t = \frac{1}{\phi_\pi} (\hat{r}_t - v_t)$$

Hence, the inflation implied by our rule is

$$\hat{\pi}_t = \sum_{i=0}^{\infty} \left(\frac{1}{\phi_\pi}\right)^{j+1} (\hat{r}_{t+j} - v_{t+j})$$

Thus, we need $v_t = \hat{r}_t$ for all t and thus, the desired rule is $i_t = \phi_\pi \pi_t + r_t$

ii. Notice that the rule can be rewritten as

$$\begin{aligned} i_t &= \phi_\pi \pi_t + \rho + \sigma \mathbb{E}_t \{\Delta y_{t+1}\} \\ &= \phi_\pi \pi_t - \sigma(1-\rho_a)\psi_{ya} a_t \\ &= \phi_\pi \pi_t - \sigma(1-\rho_a)\psi_{ya}(y_t - \psi_y) \end{aligned}$$

We know that money demand is

$$m_t - p_t = y_t - \eta i_t + \varepsilon_t^m$$

Since $\pi_t = \pi^*$ and $p_t = p_0 + \pi^* t$ in eqbm, we get

$$m_t = p_0 - \eta[\sigma(1-\rho_a)\psi_{ya}\psi_y + \phi_\pi \pi^*] + \pi^* t + [1 + \eta\sigma(1-\rho_a)\psi_{ya}]y_t + \varepsilon_t^m$$

iii. Starting from the money demand equation again.

$$m_t - p_t = y_t - \eta i_t + \varepsilon_t^m$$

replacing the fisher equation,

$$m_t - p_t = y_t - \eta \left[\mathbb{E}_t p_{t+1} - p_t + r_t\right] + \varepsilon_t^m$$

The last implies,

$$p_t = \frac{\eta}{1+\eta}\mathbb{E}_t p_{t+1} + \frac{1}{1+\eta}\left(m_t - \varepsilon_t^m\right) + u_t$$

where $u_t \equiv \eta r_t - y_t$.

Taking first differences

$$\pi_t = \frac{\eta}{1+\eta}\mathbb{E}_t\pi_{t+1} + \frac{1}{1+\eta}\left(\Delta m_t - \Delta\varepsilon_t^m\right) + \Delta u_t$$

Imposing the proposed rule $\Delta m_t = \pi^*$,

$$\pi_t = \pi^* + \mathbb{E}_t \sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \left(\Delta u_{t+k} - \frac{1}{1+\eta}\Delta\varepsilon_{t+k}^m\right)$$

Letting $\omega_t = \mathbb{E}_t \sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k \Delta u_{t+k}$ and since $\mathbb{E}_t\varepsilon_{t+k} = 0$ for all $k > 0$,

$$\pi_t = \pi^* + \omega_t - \frac{1}{1+\eta}\Delta\varepsilon_t^m$$

- Inflation will be affected by shifts in money demand because of
    - ε_t^m or changes in real variables affecting ω_t

2.3. Equilibrium indeterminacy and interest rate rules

Consider a classical economy with an exogenous real interest rate process $\{r_t\}$. Discuss the conditions under which the equilibrium will be (locally) unique and solve for the equilibrium level of inflation when the central bank follows the rules:

a. Rule #1: Partial adjustment

$$i_t = \phi_i i_{t-1} + (1 - \phi_i)\, i_t^*$$

where $\phi_i \in [0, 1]$ and i_t^* is a reference interest rate given by $i_t^* = \rho + \phi_\pi \pi_t$

b. Rule $\#2$: Moving average inflation targeting $i_t = \rho + \phi_\pi \bar{\pi}_t$ where

$$\bar{\pi}_t = (1 - \delta) \sum_{k=0}^{\infty} \delta^k \pi_{t-k}$$

c. Show the equivalence between the two rules

2.3 Answer

a. Since $(1-\phi_i L)i_t = (1-\phi_i)i_t^*$, we get $(1-\phi_i L)(r_t + \mathbb{E}_t\pi_{t+1}) = (1-\phi_i)(\rho + \phi_\pi \pi)$

That is,

$$\pi_t = \left[1 - \frac{1}{\phi_\pi}\left(\frac{1-\phi_i L}{1-\phi_i}\right)L^{-1}\right]^{-1}\left[\frac{1}{\phi_\pi}\left(\frac{1-\phi_i L}{1-\phi_i}\right)\right]\hat{r}_t$$

b. Since $\rho + \phi_\pi \bar{\pi}_t = r_t + \mathbb{E}_t\pi_{t+1}$, we get $(1-\delta)(1-\delta L)^{-1}\pi_t = \frac{1}{\phi_\pi}\hat{r}_t + \frac{1}{\phi_\pi}L^{-1}\pi_t$

That is,

$$(1-\delta)(1-\delta L^{-1})\pi_t\left[1 - \frac{1}{\phi_\pi}\left(\frac{1-\delta L}{1-\delta}\right)L^{-1}\right] = \frac{1}{\phi_\pi}\hat{r}_t$$

3.3. Government purchases and sticky prices

Consider a model economy with the following equilibrium conditions. The household's log-linearized Euler equation takes the form

$$c_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left\{\pi_{t+1}\right\} - \rho\right) + \mathbb{E}_t\left\{c_{t+1}\right\}$$

where c_t is consumption, i_t is the nominal rate, and $\pi_{t+1} = p_{t+1} - p_t$ is the rate of inflation between t and $t+1$ (Note: as in the text, lowercase letters denote the logs of the original variable.)

The household's log-linearized labor supply is given by

$$w_t - p_t = \sigma c_t + \varphi n_t$$

where w_t denotes the nominal wage, p_t is the price level, and n_t is employment. Firms' technology is given by

$$y_t = n_t$$

The time between price adjustments follows a geometric distribution, which gives rise to an inflation equation

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa \tilde{y}_t$$

where $\tilde{y}_t \equiv y_t - y_t^n$ is the output gap (with y_t^n representing the natural level of output). Assume that in the absence of constraints on price adjustment firms would set a price equal to a constant markup over marginal cost given by μ (in logs).

Suppose that the government purchases a fraction τ_t of the output of each good, with τ_t varying exogenously. Government purchases are financed through lump-sum taxes.

a. Derive a log-linear version of the goods market clearing condition of the form $y_t = c_t + g_t$, where $g_t \equiv -\log(1 - \tau_t)$

b. Determine the behavior of the natural level of output y_t^n as a function of g_t and discuss the mechanism through which a fiscal expansion leads to an increase in output when prices are flexible.

c. Assume that $\{g_t\}$ follows a simple AR(1) process with autoregressive coefficient $\rho_g \in [0, 1)$. Derive the dynamic IS equation

$$\tilde{y}_t = \mathbb{E}_t \{\tilde{y}_{t+1}\} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \pi_{t+1} - r_t^n)$$

together with an expression for the natural rate r_t^n as a function of g_t

d. Determine the economy's response to an exogenous increase in g_t when the central bank follows an interest rate rule given by

$$i_t = \rho + \phi_\pi \pi_t$$

3.3 Answer

a. The mkt clearing condition is $(1-\tau_t)Y_t = C_t$, which is $y_t = c_t + g_t$ in log deviation where $g_t \equiv -\log(1-\tau_t)$

b. Starting with markup definition

$$\begin{aligned}\mu_t &= \log(1) - (w_t - p_t) \\ &= -(\sigma c_t + \varphi n_t) \\ &= -(\sigma(y_t - g_t) + \varphi y_t)\end{aligned}$$

Thus, in flexible price eqbm,

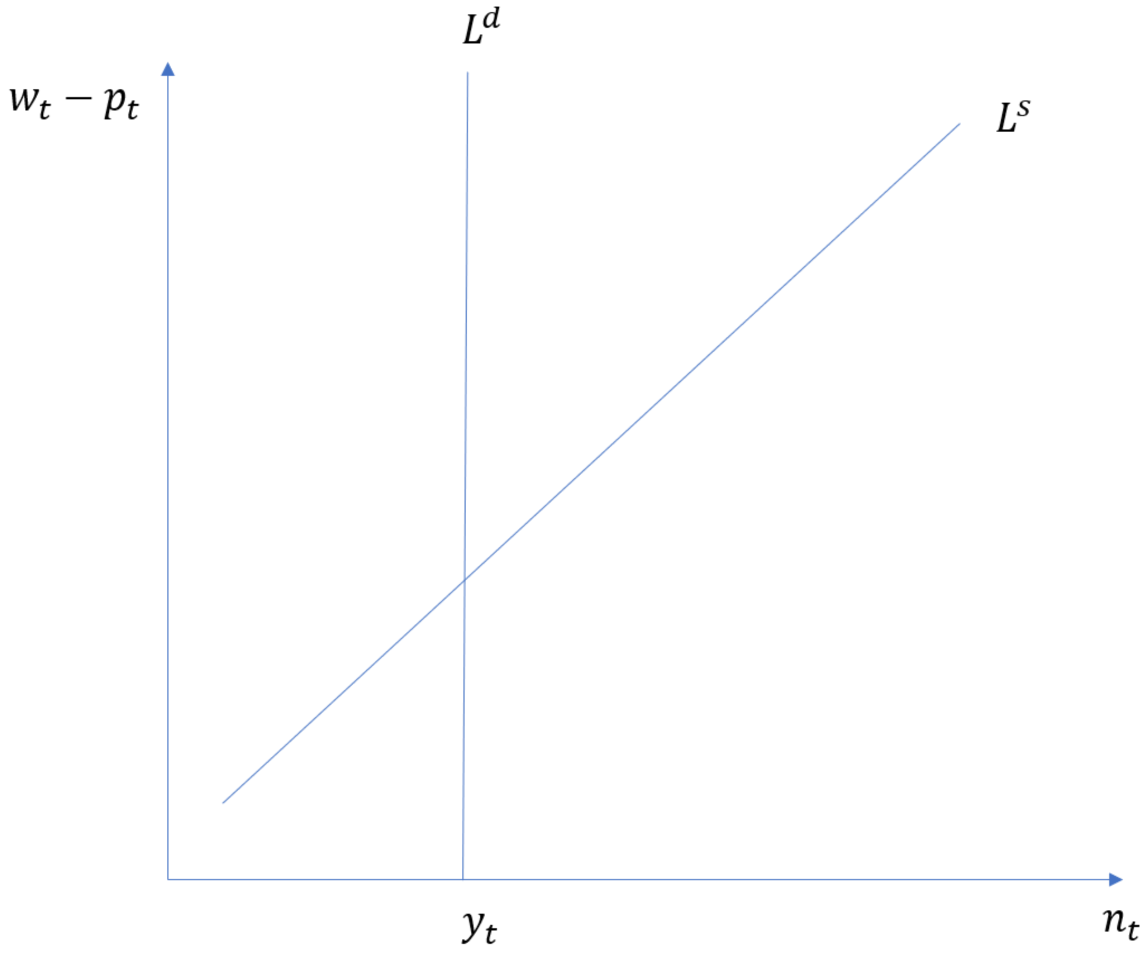
$$\begin{aligned}y_t^n &= \frac{1}{\sigma + \phi}(-\mu + \sigma g_t) \\ &= \psi_y + \psi_{yg} g_t\end{aligned}$$

Consider labor market

- $L^s : w_t - p_t = \sigma c_t + \varphi n_t$
- $L^d : n_t = y_t$

What happens when g_t goes up?

- flexible price eqbm
- sticky price eqbm



c. From the Euler equation, assuming AR(1) for g_t,

$$y_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left\{\pi_{t+1}\right\} - \rho\right) + \mathbb{E}_t\left\{y_{t+1}\right\} + (1-\rho_g)g_t$$

we get

$$\tilde{y}_t = \mathbb{E}_t\left\{\tilde{y}_{t+1}\right\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\pi_{t+1} - r_t^n\right)$$

where

$$\begin{aligned} r_t^n &= \rho + \sigma\mathbb{E}_t\Delta y_{t+1}^n - \sigma(1-\rho_g)g_t \\ &= \rho - \sigma(1+\psi_{yg})(1-\rho_g)g_t \end{aligned}$$

d. Let's check this in `gali3_3.mod`

3.7. Technology shocks and the Taylor rule

Consider an economy with Calvo-type staggered price setting, where a continuum of monopolistically competitive firms have access to a technology $Y_t = A_t N_t$, where Y_t is output, N_t denotes hours of work, and A_t is an exogenous technology parameter.

The representative consumer has a period utility $U(C_t, N) = \log C_t - \frac{N_t^{1+\varphi}}{1+\varphi}$, where C_t is a CES function of the quantities consumed of the different types of goods.

The technology parameter $a_t \equiv \log A_t$ is assumed to follow a random walk process, i.e. $a_t = a_{t+1} + \varepsilon_t$, where $\{\varepsilon_t\}$ is white noise.

Firms' desired markups are constant. All output is consumed. The labor market is perfectly competitive.

The implied equilibrium conditions take the form

$$\tilde{y}_t = \mathbb{E}_t\left\{\tilde{y}_{t+1}\right\} - \left(i_t - \mathbb{E}_t\left\{\pi_{t+1}\right\} - \rho\right)$$
$$\pi_t = \beta\mathbb{E}_t\left\{\pi_{t+1}\right\} + \kappa\tilde{y}_t$$

where $\tilde{y}_t = y_t - y_t^n$ is the output gap, y_t is $(\log)$ output, y_t^n is the (log) natural level of output, $\pi_t \equiv p_t - p_{t-1}$ is the rate of inflation, and i_t is the short-term nominal rate.

a. Determine the natural level of output y_t^n as a function of the technology parameter.

b. Suppose that the monetary authority adopts a simple interest rate rule of the form

$$i_t = \rho + \phi_\pi \pi_t$$

where $\phi_\pi > 1$. Determine the equilibrium path of inflation, the output gap, and output.

c. How would your answer to (b) change if the technology process was instead given by $a_t = \rho_a a_{t-1} + \varepsilon_t$ with $\rho_a \in (0,1)$? Derive your result analytically and explain it intuitively.

3.7 Answer

Starting with markup, equation (18) becomes

$$
\begin{aligned}
\mu_t &= a_t - (w_t - p_t) \\
&= a_t - (c_t + \varphi n_t) \\
&= a_t - (y_t + \varphi(y_t - a_t)) \\
&= (1+\varphi)a_t - (1+\varphi)y_t
\end{aligned}
$$

which gives $y_t^n = a_t - \frac{1}{1+\varphi}\mu = a_t + \psi_y$

This implies

$$\begin{aligned} r_t^n &= \rho + \sigma \mathbb{E}_t \Delta y_{t+1}^n \\ &= \rho + \sigma \mathbb{E}_t \Delta a_{t+1} \\ &= \rho \end{aligned}$$

- When the shock hits, there is no change in output gap
  - both y_t and y_t^n response to the shock by the same amount

With serial correlation, $\mathbb{E}_t \Delta a_{t+1} \neq 0$, we get negative output gap as in textbook

- since y_t reacts less to technology shock

Let's check this in `gali3_7.mod`